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Impact of two bodies one of which is covered by a thin layer of liquid

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The paper deals with the plane unsteady problem of the collision of two rigid undeformable and shallow surfaces, one of which is covered by a thin layer of an ideal incompressible liquid. At the initial instant of time, a dry surface touches the liquid free boundary at a single point and then starts to penetrate the liquid layer. The flow region is divided into four parts: the region beneath the entering surface, the jet root, the spray jet and outer region. Inside each of those subdomains the flow patterns have their own peculiarities and are analysed separately. The matching conditions allow us to obtain the uniformly valid asymptotic solution of the original problem. The relative body motion and the characteristics of the spray jets generated under the impact are determined. The condition on the shapes of the bodies, under which the velocity of the impact of the rigid surfaces is non-zero, is derived.

1. Introduction

The plane unsteady problem of the collision of two rigid shallow surfaces one of which is covered by a liquid is considered. Examples of processes of this kind can be found in machinery engineering where impacted surfaces (for example, driving wheels) are usually covered by a thin layer of oil, and in the problem of crane operation in a dock.

Another example is connected with the natural catastrophe of a huge solid mass falling into a lake from an adjacent mountain. Dr E. Baba (1994, personal communication) reported 'About 200 years ago in Shimabara of Nagasaki (Japan) a hugh solid mass, which was a part of mountain (Mau-yama) fell into shallow bay (Ariake-bay) due to an earthquake. As a result, many people living at opposite coast of the bay were killed due to tsunami (soliton)'. The closely related problem of large water waves generated by landslides was analysed numerically by Harbitz, Pedersen & Gjevik (1993). A review of a large number of Norwegian events associated with rock slides into fjords and lakes is given by Jørstad (1968). The wave generated by the fall can be very dangerous, especially for dams and power stations. The possibility of this catastrophe has to be taken into account by designers.

A sketch of the flow is shown in figure 1. Initially the liquid is at rest and occupies a region $-h-f_1(x) < y < f_1(x)$, where $f_1(0) = 0, f_1'(0) = 0$ and $f_1(x) > 0$ where $x \neq 0$. A shallow rigid surface ($y = f_2(x)$) touches the free liquid boundary ($y = -f_1(x)$) at a single point taken as the origin of the Cartesian coordinate system xOy. At some instant of time, taken as the initial one ($t = 0$), the body begins to penetrate the liquid, the initial impact velocity being V_0. The position of the entering body at an instant t is given by $y = f_2(x) - s(t)$, where $s(t)$ is the penetration depth. We shall determine the liquid flow, its boundary geometry and the body motion up to the moment T of the contact between the solid surfaces under the following assumptions: (i) the solid

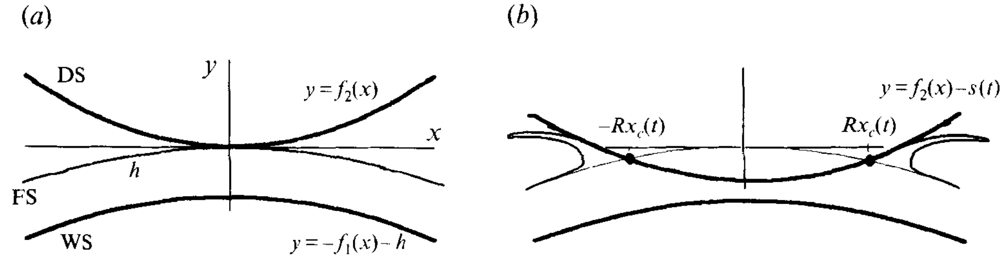

FIGURE 1. Impact of a shallow body on a liquid layer. (*a*) Initially, the liquid is at rest and occupies the region $-f_1(x)-h<y<-f_1(x)$, and the body touches the free surface at a single point: DS, dry surface; WS, wetted surface; FS, free surface of the liquid layer. (*b*) The flow pattern: $Rx_c(t)$ is the coordinate of the intersection point; $s(t)$ is the penetration depth.

surfaces are undeformable, symmetrical with respect to the y-axis and shallow, $f_1(x)/f_2(x)\to 0$ as $x\to 0$; (ii) the wetted surface is smooth, its radius of curvature at the top ($x=0$) differing from zero; (iii) the liquid is ideal and incompressible; (iv) external mass forces are potential; (v) the liquid motion is plane, symmetrical with respect to the y-axis and irrotational; (vi) surface tension is absent; and (vii) the thickness h of the liquid layer is much smaller than the dimension of the solid surfaces R.

Assumption (vii) implies that $f_j(x)=R\hat{f}_j(x/R), j=1,2$, where the $\hat{f}_j$ are dimensionless functions and $\epsilon=h/R$ is much less than unity. The coordinates of the points where the entering contour intersects the undisturbed liquid level ($y=-f_1(x)$) are $\pm Rx_c(t)$ (see figure 1*b*). The function $x_c(t)$ satisfies the equation $\hat{f}_2[x_c(t)]-s(t)/R=-\hat{f}_1[x_c(t)]$. At the moment T when the surfaces contact each other, we obtain $\hat{f}_2(x_*)+\hat{f}_1(x_*)=\epsilon$, where $x_*=x_c(T)$. It is clear that $x_*\to 0$ as $\epsilon\to 0$. We say that the solid surfaces are shallow if $x_*\gg\epsilon$. This implies that the derivative of the function $\hat{f}_2(\xi)+\hat{f}_1(\xi)$ is small at small values of the argument ξ. Moreover, assumption (i) predicts that $\hat{f}_1(x_*)=o[\hat{f}_2(x_*)]$, where $x_*\ll 1$. Therefore, at leading order as $\epsilon\to 0$, the equation for $x_*(\epsilon)$ can be approximated by a simpler one, $\hat{f}_2(x_*)=\epsilon$, which on differentiating in ϵ, yields $\hat{f}_2'(x_*)\,\mathrm{d}x_*/\mathrm{d}\epsilon=1$. Here $\mathrm{d}x_*/\mathrm{d}\epsilon$ is of the order x_*/ϵ; therefore $x_*(\epsilon)/\epsilon=O(1/\hat{f}_2'(x_*))$.

For example, if $f_1(x)\equiv 0$, $f_2(x)=(1/n)Rk(|x|/R)^n$, $s(t)=V_0t$, $n>0$, $k>0$, we obtain $x_*=(n\epsilon/k)^{1/n}$. The condition $x_*\gg\epsilon$ is satisfied when $n^{1/n}\epsilon^{(1/n)-1}k^{-1/n}\gg 1$. This is possible in the following two cases. (*a*) $n>1$, $\epsilon\ll 1$, $k=O(1)$; (*b*) $n=1$, $k\ll 1$. In the latter case the introduction of R is formal.

We take Rx_* as the lengthscale in the x-direction, and h as the lengthscale in the y-direction. It can be shown that the problem under consideration can be approximately transformed to the water-entry problem, which has a simpler geometry than the original one. The transformation $y_1=y+f_1(x)$ maps the original position of the free surface onto the horizontal line $y_1=0$ and the position of the wetted surface onto the line $y_1=-h$. The position of the entering surface is now described by the equation $y_1=f_2(x)+f_1(x)-s(t)$. It can be verified that this mapping does not change the equations of motion, or boundary or initial conditions at leading order with respect to x_*. The derivatives $\partial/\partial y$, $\partial/\partial t$ are transformed by this mapping into $\partial/\partial y_1$ and $\partial/\partial t$, respectively. The derivative $\partial/\partial x$ is transformed into

$$\frac{\partial}{\partial x}+\left(\frac{\partial f_1}{\partial x}\right)\frac{\partial}{\partial y_1}$$

or, in the dimensionless variables, into

$$\frac{1}{Rx_*}\left[\frac{\partial}{\partial(x/Rx_*)}+\frac{Rx_*}{h}\hat{f}_1'\left(\frac{x}{R}\right)\frac{\partial}{\partial(y_1/h)}\right].$$

The product $(Rx_*/h)\hat{f}_1'(x/R)$ is of the order $\epsilon^{-1}x_*(\epsilon)\hat{f}_1'(x_*)$ where $x/Rx_* = O(1)$. We have obtained above that $x_*(\epsilon)/\epsilon = O(1/\hat{f}_2'(x_*))$, which predicts

$$\frac{Rx_*}{h}\hat{f}_1'\left(\frac{x}{R}\right) = O(\hat{f}_1'(x_*)/\hat{f}_2'(x_*)).$$

Assumption (i) implies that the wetted surface is shallower than the dry one, $\hat{f}_1(x_*)/\hat{f}_2(x_*)\to 0$ as $x_*\to 0$. By L'Hospital's rule we get $\hat{f}_1'(x_*)/\hat{f}_2'(x_*)\to 0$ as $x_*\to 0$. Therefore, at leading order the mapping does not change the operators of differentiation in x, y, t. The water-impact problem ($f_1(x)\equiv 0$) is considered below; however, the results presented are valid not only for this problem but also for the more general one with both solid surfaces curved.

It is important to note that the liquid flow depends not on the whole shape of the entering body but only on the shape of the wetted part, i.e. on the body geometry near its top ($x = 0$). In a small vicinity of the top, the shape of the body may be approximated by a simpler one. If the body is blunt and smooth, the Taylor expansion may be used that yields $\hat{f}_2(\xi) = c_1\xi^2+o(\xi^2)$, $c_1 \geqslant 0$ as $|\xi| \ll 1$. The approximation of the body shape by a parabolic one is quite general (see Korobkin & Pukhnachov 1988) but it does not cover all possible cases. The analysis developed in the present paper is valid for an arbitrary blunt body. Numerical results are given for the case when $\hat{f}_2(\xi)$ can be approximated near the point $\xi = 0$ by a power function.

The impact by a pointed body, the deadrise angle of which is not small, by a cusped body, and by a flat-bottomed body are not considered here, and the impact of a box-like structure onto shallow water will be analysed in a future paper by the author. The analysis will be based on both the present approach and the experimental results by V. I. Bukreev (1994, personal communication).

The main focus of the present paper is the plane problem, but the approach suggested is valid in the three-dimensional case as well. There are three reasons for analysing the two-dimensional problem first: (i) the plane problem is much simpler than the three-dimensional one but yet contains the main peculiarities of the general impact problem; (ii) the plane problem is more suitable for experimental analysis and verification of the model; and (iii) the present approach may be combined with strip theory to obtain an approximation to the three-dimensional solution. The strip theory implies that the variation of the flow in the cross-sectional plane is much larger than the variation of the flow in the longitudinal direction. It is clear that this approximation will not be true at the ends of the body.

The impact of a shape that is symmetric about the y-axis is considered for simplicity only.

Two cases should be distinguished:

(i) Initially the body is placed above the liquid and then starts to approach the liquid free surface. The air which is pushed ahead of the body cannot escape completely from the gap between the shallow bottom and the free surface and, as a result, a cavity filled by the entrapped air may be formed at the contact instant. The air flow between the bottom and the liquid surface before the contact occurs is of great importance (Iwanowski & Yao 1992, 1993).

(ii) Initially the body touches the liquid and then starts to penetrate it. We may expect that in this case the presence of air does not influence significantly the impact because the liquid density is much greater than the density of the air. This is the case which is considered in the present paper.

The impact of a shallow-bottomed body onto a liquid layer may be divided into the following four stages:

(i) At the first stage the speed of the contact region expansion is of the order of the sound velocity in the resting liquid c_0 (Korobkin 1992). The duration of this stage is small and can be estimated from the relation $R(\mathrm{d}x_c/\mathrm{d}t)(t) = O(c_0)$.

(ii) At the second stage the acoustic effects may be disregarded, and the dimension of the contact region is much smaller than the layer depth h. However, the depth of body penetration is much smaller than the contact region size. Thus the presence of the bottom may be neglected and the liquid layer may be changed for the lower half-plane. To estimate the duration of this stage, the following relations may be used: $s(t) \ll Rx_c(t) \ll h$. This case was analysed by Wagner (1932), who suggested approximating the wetted part of the entering body by a plate of width unknown in advance and so must be determined together with the liquid flow. The Wagner approach is now the main tool in ship hydrodynamics analyses.

(iii) At the third stage the depth of body penetration is much smaller than the thickness of the liquid layer, $s(t) \ll h$, but the dimension of the contact region is of the same order, $Rx_c(t) = O(h)$. This case was analysed by Galanin & Saikin (1981) within the framework of the Wagner approach.

(iv) At the fourth stage the depth of body penetration is comparable with the layer thickness, $s(t) = O(h)$, but the dimension of the contact region is much larger than it, $Rx_c(t) \gg h$. This is the stage which is under consideration in the present paper.

Four important remarks should be made:

(i) For a given shape of entering body some stages may be absent. For example, in the wedge-entry problem there is only the first stage when the deadrise angle is of $O(V_0/c_0)$.

(ii) In order to determine the characteristics of each stage, the dimension of the wetted part of the entering body was used above. This dimension is unknown in advance at each stage, and it has to be determined together with the liquid flow and the pressure distribution. But to estimate the duration of each stage, the function $x_c(t)$ which is found in advance from purely geometrical reasonings and which is of the same order as the dimension of the contact region may be used. That is why the relations given above can be used for preliminary estimations but not for calculations.

(iii) The duration of each stage under consideration is much larger than the durations of the previous stages. This can be verified using the relations mentioned above. Therefore, at leading order every stage may be considered separately without any reference to the previous stages.

(iv) At each stage, except the fourth, the deformations of the liquid domain due to the impact are small compared with the domain size. Therefore, under assumptions (i)–(vii) the equations of motion and the boundary conditions may be linearized, and the boundary conditions may be taken at the undisturbed liquid level in the first approximation. This idea was used by Korobkin (1994) for the first stage, by Wagner (1932) for the second, and by Galanin & Saikin (1981) for the third stage. At the fourth stage the deformation of the liquid domain must be taken into account in any case. That is why the present analysis may be used to test numerical schemes developed to describe flows with large deformations.

2. Formulation of the problem

The entry of a blunt contour into a layer of ideal incompressible liquid is considered below (see figure 2). At the initial instant of time ($t = 0$), the body touches the liquid free surface at a single point taken as the origin of the Cartesian coordinate system xOy. The liquid initially is at rest and occupies the strip $-h < y < 0$. The line $y = 0$ corresponds to the undisturbed position of the free liquid surface, and the line $y = -h$ to the rigid bottom. Then the body starts to penetrate the liquid layer at an initial impact velocity of V_0. The position of the entering body is given by the equation $y = f(x) - s(t)$ where $s(t)$ is the depth of penetration. It is necessary to determine the liquid flow and the body motion under assumptions (i)–(vii) of the previous section.

The liquid motion is governed by the Euler equations

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}, \tag{1}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}-g, \tag{2}$$

with respect to the velocity vector of liquid particles $\boldsymbol{u}(x, t) = (u, v)$ and the pressure $p(x, y, t)$; ρ is the liquid density, g is the acceleration due to gravity. The equation of continuity for an incompressible liquid is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0. \tag{3}$$

The liquid flow is assumed irrotational; therefore, the equation

$$\frac{\partial u}{\partial y}=\frac{\partial v}{\partial x} \tag{4}$$

is satisfied. Let $F_s(x, y, t) = 0$ describe the position of the free surface. On the free surface, the position of which is unknown in advance, the kinematic condition

$$\frac{\partial F_s}{\partial t}+u\frac{\partial F_s}{\partial x}+v\frac{\partial F_s}{\partial y}=0 \tag{5}$$

and the dynamic condition

$$p = 0 \tag{6}$$

hold. On the wetted part of the entering contour, the size and position of which will be determined together with the problem solution, the normal component of the velocity of the body and that of the liquid particles are equal:

$$v = f'(x)\,u - s'(t) \quad (y = f(x) - s(t),\ |x| < c(t)). \tag{7}$$

Here $c(t)$ is one half the dimension of the wetted part of the entering body (see figure 2b). On the bottom, $y = -h$, the vertical component of the liquid velocity $v(x, -h, t)$ is zero:

$$v = 0 \quad (y = -h,\ -\infty < x < +\infty). \tag{8}$$

The initial conditions are

$$\boldsymbol{u} = 0,\quad p = 0,\quad c = 0,\quad s = 0,\quad s' = V_0,\quad F_s(x, y, 0) = y \quad (t = 0). \tag{9}$$

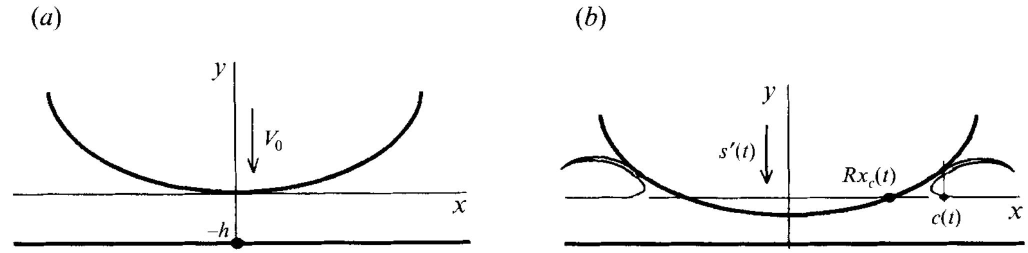

FIGURE 2. Impact of a solid body on the free surface of a liquid layer. (*a*) Initially, the liquid is at rest and occupies the strip $-h < y < 0$ and the body touches the free surface at a single point. (*b*) The flow pattern: $2c(t)$ is the dimension of the contact region.

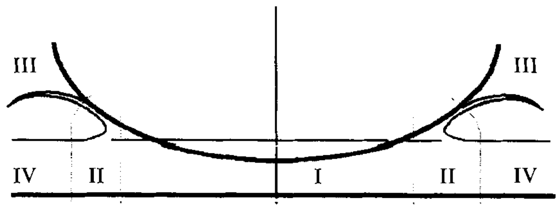

FIGURE 3. Sketch of the liquid flow: I, the region beneath the body; II, the jet roots; III, the spray jets; IV, the outer region.

We assume that near the top, $x = 0$, of the entering contour, the body shape may be approximated by a power function:

$$f(x) = \frac{1}{n} Rk \left(\frac{|x|}{R}\right)^n. \tag{10}$$

Assumption (vii) will be satisfied in the following two cases (see §1): (i) $n > 1$, $h/R \ll 1$; (ii) $n = 1$, $k \ll 1$. In the first case R may be defined in such a way that $k = 1$. In the second case R is a formal parameter, and it is convenient to take $R = h/k$. A body with $n = 1$ is a wedge, and bodies with $n > 1$ have a flatter bottom than a wedge, i.e. are locally horizontal at their vertex. In both cases there is a small parameter $\epsilon = h/R$ in the problem (1)–(9). We shall determine an approximate solution of the problem (1)–(10), which is uniformly valid as $\epsilon \to 0$ up to the moment of contact between the entering body and the bottom, $0 < t < T$.

In order to construct an approximate solution, the method of matched asymptotic expansions is used. In accordance with this method the flow is divided into the four regions shown in figure 3: I, the region beneath the entering body; II, the jet root; III, the spray jet; IV, the outer region.

In region I, the orders of the independent variables and the unknown functions are as follows: $x = O(Rx_*)$, $y = O(h)$, $t = O(h/V_0)$, $v = O(V_0)$, (3) gives that $u = O(V_0 Rx_*/h)$, (1) gives $p = O(\rho V_0^2 [Rx_*/h]^2)$, (4) gives $\partial u/\partial y = O(V_0/Rx_*)$. Here $x_* = x_c(T)$ and $x_* = (n\epsilon/k)^{1/n}$ when (10) is valid (see figure 2*b*). It was mentioned above that $x_*(\epsilon) \to 0$ and $\epsilon x_*^{-1}(\epsilon) \to 0$ as $\epsilon \to 0$. Therefore, the term $v\,\partial u/\partial y$ in (1) is much smaller than other terms in the equation, and it may be omitted at leading order as $\epsilon \to 0$. Accordingly, all terms in (2) are much smaller than the term $(1/\rho)(\partial p/\partial y)$. This means that in this region, $\partial p/\partial y = 0$ at leading order. Equation (4) gives at leading order $\partial u/\partial y = 0$. These estimates make it possible to consider the pressure p and the horizontal component of the velocity u as approximately y-independent. Equations (3), (7)–(9) do not change their forms when $\epsilon \to 0$.

The characteristic of the problem considered is that one has to present the flow scheme in advance in a way which will allow us to construct the uniformly valid combined solution, which is possible if the flow in the region II is essentially two-dimensional. This region moves from the centreline at velocity $c'(t)$. The internal variables λ, μ which have to be introduced in the region are $x = c(t)+\lambda$, $y = \mu$, where $\lambda = O(h)$, $\mu = O(h)$. The horizontal velocities of the liquid particles in regions I and II must be comparable, which gives $u = O(V_0 Rx_*/h)$. Then (3) yields $v = O(V_0 Rx_*/h)$, and (7) shows that the velocity of the body $s'(t)$ inside the jet root may be neglected compared with the flow velocity. The derivative in time $\partial/\partial t$ is transformed in the internal variables into the operator $\partial/\partial t - c'(t)\partial/\partial\lambda$ which may be rewritten in dimensionless variables in the form

$$\frac{V_0}{h}\left(\frac{\partial}{\partial(tV_0/h)} - \frac{c'(t)}{V_0}\frac{\partial}{\partial(\lambda/h)}\right).$$

But $c'(t)/V_0 = O((R/V_0)x_c'(t))$, where $x_c(t)$ satisfies the equation $f[Rx_c(t)] = s(t)$. Differentiating this equation in time, one obtains $Rx_c'(t) = s'(t)/f'[Rx_c(t)]$. Therefore, $c'(t)/V_0 = O(1/f'(Rx_*))$ and is much greater than unity. Equation (10) predicts $f'(Rx_*) = n\epsilon x_*^{-1}(\epsilon)$, which is a small quantity. Moreover, it can be verified that $c'(t) = O(V_0 Rx_*/h)$, i.e. the speed of the contact region (region I) expansion is of the order of the liquid velocity in region II. The analysis presented indicates that the derivatives in time, $\partial/\partial t$, in (1)–(9) can be substituted by $-c'(t)\partial/\partial\lambda$ with accuracy up to $O(f'(Rx_*))$. This means that the flow in region II may be considered as approximately quasi-stationary at leading order as $\epsilon \to 0$.

In the jet region (region III) the pressure is near the atmospheric value and, hence, the liquid particles in the jet move inertially and tangentially to the entering surface. The flow inside the jet region was analysed by Howison, Ockenden & Wilson (1991) within the framework of the classical Wagner theory. It was shown that the flow is approximately one-dimensional and depends on that in the jet root; the influence of the jet motion on the flow inside the jet root may be neglected.

The critical velocity for the liquid layer is equal to $(gh)^{1/2}$ where g is the acceleration due to gravity. When $\mathrm{d}c/\mathrm{d}t > (gh)^{1/2}$, the liquid outside the contact region, i.e. in region IV, remains at rest. In order to find the condition under which the liquid in region IV will be at rest for $0 < t < T$, we can put that $c'(t) = O(Rx_c'(t))$. But

$$R(\mathrm{d}x_c/\mathrm{d}t)(t) > R(\mathrm{d}x_c/\mathrm{d}t)(T) = V_0 k^{-1/n}(n\epsilon)^{(1/n)-1}$$

and the condition $\mathrm{d}c/\mathrm{d}t > (gh)^{1/2}$ can be rewritten as $V_0/(gh)^{1/2} \gg n\epsilon x_*^{-1}(\epsilon)$. When the velocity of the contact region expansion approaches the critical value, a soliton may be formed. Further, the soliton escapes from the entering surface and propagates at a velocity which is about the critical value.

The viscous effects may be neglected at leading order everywhere when $V_0 \gg \nu/R$ where ν is the kinematic viscosity coefficient ($\nu = 0.01$ cm^2 s^{-1} for water). When $R =$ 50 m, we get the inequality $V_0 \gg 2\times 10^{-8}$ m s^{-1}.

We shall determine the flow characteristics in regions I and II, and then match them with each other and with the rest state in region IV. It will be shown that this procedure makes it possible to find approximately all the characteristics of the liquid flow and the body motion up to the moment of the contact of the entering body with the bottom.

3. The liquid flow between the entering surface and the bottom

The asymptotic analysis and physical reasoning indicate that in region I the equations of motion (1)–(4) can be simplified and taken in the forms

$$u_t + uu_x = (1/\rho)p_x, \tag{11}$$

$$u_x + v_y = 0 \quad (|x| < c(t), -h < y < f(x) - s(t)), \tag{12}$$

where $u = u(x, t)$, $p = p(x, t)$, $v = v(x, y, t)$. The boundary conditions (7), (8) give

$$v = f'(x)u - s'(t) \quad (y = f(x) - s(t), |x| < c(t)), \tag{13}$$

$$v = 0 \quad (y = -h, |x| < c(t)). \tag{14}$$

Integrating (12) with respect to y and taking (13) and (14) into account, we find

$$u(x, t) = \frac{s'(t)\,x}{f(x) + h - s(t)} \tag{15}$$

in the symmetrical case. (The prime denotes a derivative.) The values of the functions at $x = c(t)$ will be denoted by the subscript c. Then

$$u_c(t) = \frac{s'(t)\,c}{H_c(t)}, \tag{16}$$

where $H_c(t) = f(c) + h - s(t)$ is the layer thickness at the boundary of region I. In (15) and (16) the functions $c(t)$, $s(t)$ remain unknown.

In order to find the pressure distribution over the contact spot, we have to integrate (11) using (15) and the boundary condition $p(c(t), t) = p_c(t)$, where $p_c(t)$ is unknown in advance. We get, after integration,

$$p(x, t) = p_c(t) + \tfrac{1}{2}\rho[u_c^2(t) - u^2(x, t)] + \rho \int_x^c u_t(\xi, t)\,\mathrm{d}\xi. \tag{17}$$

The hydrodynamic force $F(t)$ on the entering body is

$$F(t) = \int_{-c}^{c} p(x, t)\,\mathrm{d}x = (2p_c(t) + \rho u_c^2(t))\,c(t)$$
$$+ 2\rho s''(t) \int_0^c \frac{x^2\,\mathrm{d}x}{f(x) + h - s(t)} + \rho s'^2 \int_0^c \frac{x^2\,\mathrm{d}x}{[f(x) + h - s(t)]^2}. \tag{18}$$

The second Newton law yields $m_B s'' = m_B g - F(t)$, where m_B is the mass of the entering body per unit length. Substitution of (18) into this equation leads to the differential equation with respect to the function $s = s(t)$:

$$A(c, s)\,s'' + B(c, s)\,(s')^2 = C(p_c, c), \tag{19}$$

where

$$A(c, s) = m_B + 2\rho \int_0^{c(t)} \frac{x^2\,\mathrm{d}x}{f(x) + h - s(t)}, \tag{20}$$

$$B(c, s) = \rho\left[\int_0^{c(t)} \frac{x^2\,\mathrm{d}x}{[f(x) + h - s(t)]^2} + \frac{c^3(t)}{H_c^2(t)}\right], \tag{21}$$

$$C(p_c, c) = m_B g - 2p_c(t)\,c(t). \tag{22}$$

The initial conditions for (19) are

$$s(0) = 0, \quad s'(0) = V_0 \quad (t = 0). \tag{23}$$

In problems of machinery engineering the gravity force must be omitted.

Thus, the flow and the pressure distribution in region I are determined with the help of two arbitrary functions $c(t)$, $p_c(t)$. To find them, we need to consider the flow in region II.

4. Liquid flow in region II and matching conditions

In the moving coordinate system which translates to the right at the velocity $\mathrm{d}c/\mathrm{d}t$, the flow in region II can be considered as approximately quasi-stationary, the entering body velocity can be neglected and the body surface can be taken as a horizontal plate (figure 4). Within the framework of this scheme the jet of the thickness h moves left with the velocity $\mathrm{d}c/\mathrm{d}t$. A part of the jet mass continues to move left between the two rigid horizontal plates, the distance between which being $H_c(t)$. The pressure is $p_c(t)$ and the horizontal velocity is $u_0(t)$ at the left-hand-side infinity. Another part of the jet is turned and forms a spray jet with thickness $h_j(t)$. The dynamical condition on the free surface demands that in the quasi-stationary case the magnitude of the flow velocity on the free surface is constant. Therefore, the horizontal velocity of the spray jet at infinity is $\mathrm{d}c/\mathrm{d}t$ (see Tuck & Dixon 1989). We have used here the conditions of matching the flow parameters in regions I and II. Moreover, these conditions give $u_0(t) = \mathrm{d}c/\mathrm{d}t - u_c(t)$.

This problem was solved by Tuck & Dixon (1989). But if we are not interested in a detailed analysis of the fine flow structure, the integral conservation laws can be used to find the relations between the flow characteristics away from region II. We obtain the following relations:

mass conservation law

$$hc' = h_j c' + H_c u_0, \tag{24}$$

Bernoulli's equation (which is equivalent to the energy conservation law)

$$\tfrac{1}{2}c'^2 = \frac{1}{\rho}p_c + \tfrac{1}{2}u_0^2, \tag{25}$$

momentum conservation law

$$(p_c + \rho u_0^2)\, H_c = \rho c'^2 (h + h_j). \tag{26}$$

Here

$$u_0(t) = c'(t) - cs'/H_c. \tag{27}$$

We have the four equations (9)–(12) for the unknown functions $h_j(t)$, $c(t)$, $p_c(t)$, $u_0(t)$. By algebra we find

$$\frac{\mathrm{d}c}{\mathrm{d}t} = \frac{cs'}{2H_c[1-(h/H_c)^{1/2}]}, \tag{28}$$

$$p_c(t) = \frac{\rho c^2 s'^2}{2H_c^2[(H_c/h)^{1/2} - 1]}, \tag{29}$$

$$h_j = h[(H_c/h)^{1/2} - 1]^2. \tag{30}$$

Equations (28)–(30) make it possible to define the flow in region I and the entering body motion.

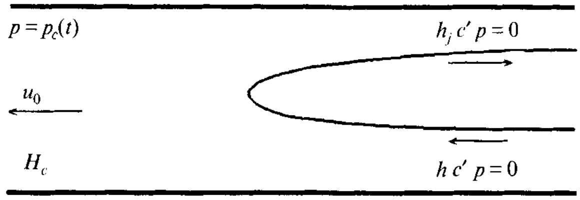

FIGURE 4. The flow pattern in the jet root region.

5. Motion of the body

The differential equation (28) can be rewritten as

$$\frac{\mathrm{d}c}{\mathrm{d}s}=\frac{c}{2H_c[1-(h/H_c)^{1/2}]} \quad (0<s<h), \tag{31}$$

where the right-hand side is dependent on c and s only. This means that (31) with the initial condition

$$c=0 \quad (s=0) \tag{32}$$

defines the size of the contact region $2c$ as a function of the penetration depth s only. Inserting (29) in the equation of the body motion (19), we obtain

$$A_1(s)\,s''+A_2(s)\,(s')^2=m_B\,g\,(0<t<T), \tag{33}$$

$$s(0)=0, \quad s'(0)=V_0, \tag{34}$$

where

$$A_1(s)=A(c(s),s), \quad A_2(s)=\rho\int_0^{c(s)}\frac{x^2\,\mathrm{d}x}{(f(x)-s+h)^2}+\frac{\rho c^3(s)}{H_c^2[1-(h/H_c)^{1/2}]}. \tag{35}$$

The value of T is unknown in advance and has to be found from the condition $s(T)=h$. Thus, the motion of the body will be found after solving the initial-value problems (31), (32) and (33)–(35).

These problems can only be solved numerically, but the analysis of their solutions is necessary. It is convenient to introduce the non-dimensional depth of penetration $\sigma=s/h$ and the new unknown function $U(\sigma)=H_c/h-1$. Then hU is the elevation of the liquid at the point $x=c(t)$, and the formula (30) for the jet thickness may be rewritten as

$$h_j=h([1+U(\sigma)]^{1/2}-1)^2. \tag{36}$$

In the case $f(x)=Rkn^{-1}(|x|/R)^n$ (see §2), one can verify that

$$c(\sigma)=R\left(\frac{nh}{kR}\right)^{1/n}(U(\sigma)+\sigma)^{1/n} \tag{37}$$

and (31) leads to a simple differential equation for $U(\sigma)$:

$$\frac{\mathrm{d}U}{\mathrm{d}\sigma}=\frac{n}{2}\left(1+\frac{\sigma}{U}\right)(1+[1+U]^{-1/2})-1 \quad (0<\sigma<1), \tag{38}$$

$$U=0 \quad (\sigma=0). \tag{39}$$

It is important that (38) and (39) depend on the only parameter n. We obtain

$$U(\sigma)\approx n\sigma-\frac{n^2(n+1)}{4(2n+1)}\sigma^2-\frac{n^3(n+1)(7n^2+8n+2)}{8(2n+1)^2(3n+1)}\sigma^3+O(\sigma^4) \tag{40}$$

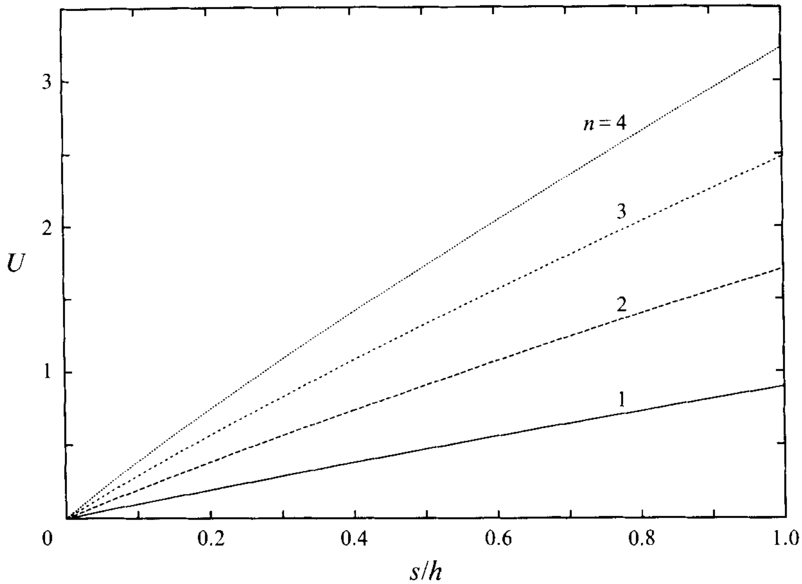

FIGURE 5. Graph of the function $U(s/h)$ for different values of the parameter n.

as $\sigma \rightarrow 0$. The asymptotic equation (40) is necessary to start the numerical solution of (38) and (39). The problem (38)–(40) was solved numerically by the Runge–Kutta method with step size 0.01. At the first and second steps the values of $U(\sigma)$ were calculated using the asymptotic equation (40). Halving the step size changes the result by less than 2×10^{-7}. The function $U(\sigma)$ monotonically increases with σ and n (figure 5). All other unknown quantities can be expressed in quadratures with the help of this function.

Let us find the function $t(\sigma)$ and its first derivative $t'(\sigma)$. Then the characteristics of both the liquid flow and the body motion will be defined in parametrical form with the dimensionless depth of penetration σ as the parameter. In order to find the derivative $t'(\sigma)$, let us consider the initial problem (33)–(35), the solution of which will be sought in the form $s' = [L(s)]^{1/2}$. The new function $L(s)$ satisfies the equation

$$A_1(s)\, L' + 2A_2(s)\, L = 2m_B\, g \tag{41}$$

and the initial condition

$$L(0) = V_0^2. \tag{42}$$

The solution of the initial-value problem (41) and (42) provides the dependence of the body velocity s' on the penetration depth s. Let us introduce the non-dimensional time $\tau = tV_0/h$, then

$$\frac{\mathrm{d}\tau}{\mathrm{d}\sigma} = \Gamma^{-1}(\sigma)\left(1 + 2Fr^{-2}\int_0^{\sigma}\frac{\mathrm{d}z}{E(z)\,\Gamma^2(z)}\right)^{-1/2}, \tag{43}$$

$$\Gamma(\sigma) = E^{-1/2}(\sigma)\exp\left(-\mu\int_0^{\sigma}\frac{(U(z)+z)^{3/n}\,\mathrm{d}z}{E(z)\,(U(z)+1)^2\,(1-(U(z)+1)^{-1/2})}\right), \tag{44}$$

$$E(\sigma) = 1 + 4\mu(U(\sigma)+\sigma)^{3/n}\int_0^1\frac{z^2\,\mathrm{d}z}{z^n(U(\sigma)+\sigma)+1-\sigma}, \tag{45}$$

$$\mu = \frac{1}{2}\frac{\rho R^2}{m_B}\left(\frac{R}{h}\right)^{1-3/n}\left(\frac{n}{k}\right)^{3/n}, \quad Fr = \frac{V_0}{(gh)^{1/2}}. \tag{46}$$

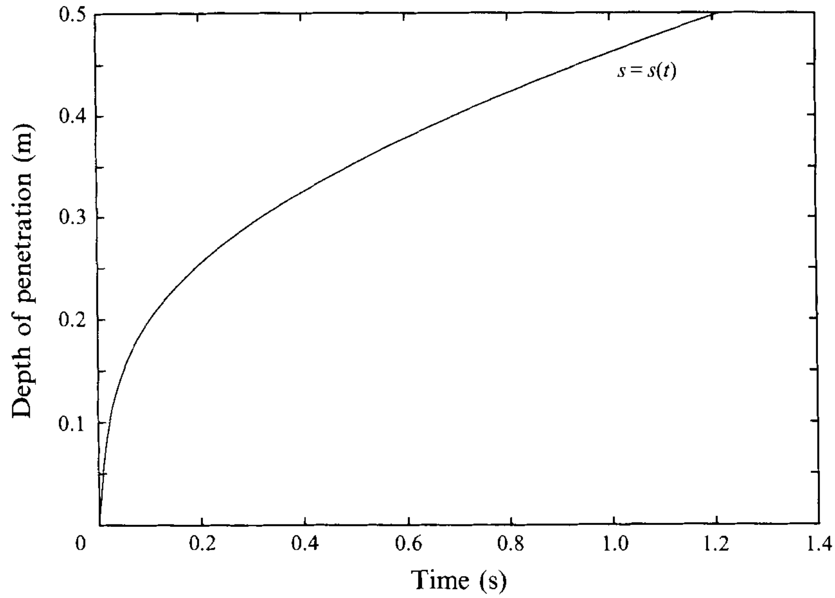

FIGURE 6. The depth of the wedge penetration as function of time.

Equations (43)–(46) give the solution in quadratures and allow its analysis in detail. Note that two impact processes will be mechanically similar only when the geometrical parameter n, the statical parameter μ and the Froude number Fr have the same value.

In the case of a wedge impact ($n = 1$), we get

$$\mu = \frac{\rho h^2}{m_B k^3}, \tag{47}$$

where k is small owing to assumption (vii). If the wedge is heavy ($\mu \ll 1$), equations (43)–(47) indicate that $\Gamma(\sigma) \approx 1$ and the wedge hits the bottom at the velocity $V_0(1+2Fr^{-2})^{1/2}$. The corresponding velocity for a light wedge ($\mu \gg 1$) is of $O(V\mu^{-1/2})$, but its dependence on time is non-trivial.

The numerical calculations were done for a wedge $y = 0.1\,|x|$ of mass $m_B = 7000$ kg m^{-1} entering a water layer of depth 0.5 m at initial velocity $V_0 = 6$ m s^{-1} ($\mu = 35.7$, $Fr = 2.7$, $n = 1$). The integrals in (43)–(45) were evaluated by Simpson's rule with the fixed step size 0.01. All integrands are bounded and smooth functions of their arguments. This case was used not for simplicity of calculation but to demonstrate the typical evolution of the process. Figure 6 shows that the duration of the wedge penetration is relatively large, approximately 1.2 s. Initially the entry velocity grows a little owing to the gravity force, and then quickly vanishes (figure 7). At the final stage the variations of the body velocity are quite small up to the moment of the collision with the bottom. It is seen that the velocity of the contact region expansion $\mathrm{d}c/\mathrm{d}t$ remains much greater than the body velocity and greater than the critical velocity $(gh)^{1/2}$ which is equal to 2.215 m s^{-1} in this example. The jet thickness h_j as function of the penetration depth s is shown in figure 8.

The body motion near the bottom is of particular interest. We shall find the asymptotic behaviour of the body velocity $s'(t)$ and time of penetration $t(\sigma)$ as $\sigma \to 1-0$. We assume that $\mu = O(1)$ and will denote $U(1)+1$ by a. It is seen that for $n < 3$ all functions in (43)–(45) are bounded. Hence, in this case the time of penetration $T = t(1)$

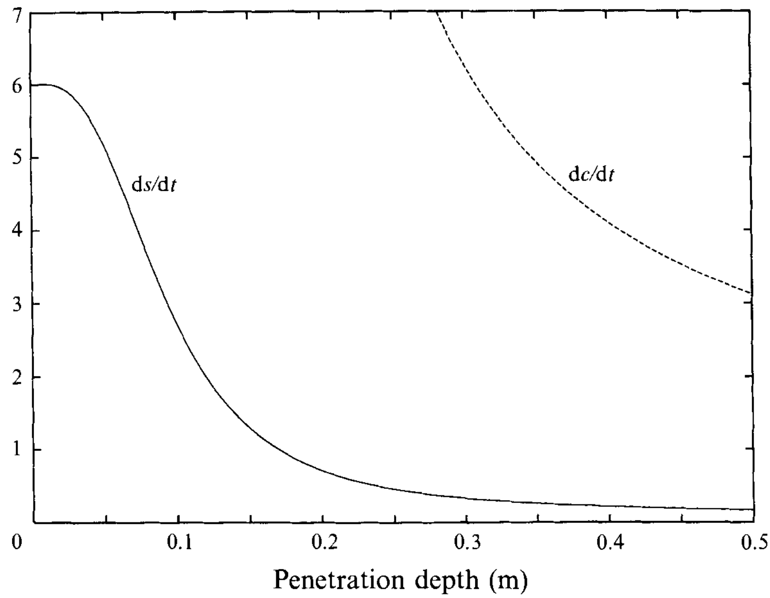

FIGURE 7. Velocity of the wedge (——) and velocity of the contact region expansion (-----) as functions of the penetration depth s. Initial velocity of the wedge is 6 m s^{-1}, the liquid depth is 0.5 m.

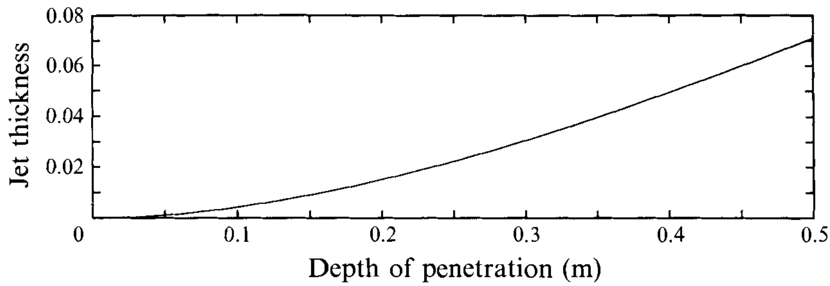

FIGURE 8. Thickness of the jet h_j as function of the penetration depth s. Maximum thickness is 7 cm.

and the velocity of the collision between the moving body and the bottom $s'(T)$ are finite. When $n = 3$, we obtain as $\sigma \to 1\text{–}0$

$$E(\sigma) \approx -\tfrac{4}{3}\mu \ln(1-\sigma), \quad \Gamma(\sigma) \approx C_1 \ln^{-1/2}\left(\frac{1}{1-\sigma}\right), \tag{48}$$

$$\frac{\mathrm{d}s}{\mathrm{d}t} = O\left[\ln^{-1/2}\left(\frac{1}{1-\sigma}\right)\right], \tag{49}$$

where the constant C_1 depends on the motion history. Equality (49) means that the body approaches the bottom at zero velocity; (48) gives that the value of T is finite and can be calculated. We $n > 3$, we find

$$E(\sigma) \approx \mu\zeta(a)(1-\sigma)^{(3/n)-1}, \quad \zeta(a) = 4a^{3/n}\int_0^\infty \frac{\xi^2\,\mathrm{d}\xi}{a\xi^n+1}, \tag{50}$$

$$\Gamma(\sigma) \approx C_2(1-\sigma)^{(1/2)-(3/2n)}, \quad \frac{\mathrm{d}s}{\mathrm{d}t} = O[(1-\sigma)^{(1/2)-(3/2n)}], \tag{51}$$

where C_2 is a constant. Therefore, in this case also the collision velocity $s'(T)$ is zero and T is finite.

6. Validity of the scheme

The result that $s'(T) = 0$ for $n \geqslant 3$ is not correct, because just before the moment T the present scheme for the flow fails. In order to demonstrate this point, let us find the asymptotic behaviour of the speed of the contact region expansion $\mathrm{d}c/\mathrm{d}t$ as $\sigma \to 1$. Taking (37), (43)–(45) into account, we obtain

$$\frac{\mathrm{d}c}{\mathrm{d}t} \approx \frac{V_0}{2}\frac{a^{(1/n)-(1/2)}}{a^{1/2}-1}\left(\frac{n}{k}\right)^{1/n}\left(\frac{R}{h}\right)^{1-1/n}(1-\sigma)^{(1/2)-(3/2n)}, \tag{52}$$

where $n > 3$ and $\sigma \to 1$. It is seen from (52) that the velocity $\mathrm{d}c/\mathrm{d}t$ is less than the critical value $(gh)^{1/2}$ just before the contact of the body with the bottom occurs. This means that a soliton may be formed at the end of the body motion.

Another limit on the scheme validity is connected with the flow velocities and the pressure distribution beneath the body, which can be very high. The flow velocity between the body and the bottom is defined by (15). The velocity $u(x,t)$ increases monotonically from zero at $x = 0$ to $u_c(t)$ at $x = c(t)$ up to the moment when $(n-1)\,U(\sigma) = 1-n\sigma$. For a parabolic shape ($n = 2$) at this moment the penetration depth is approximately a quarter of the layer depth (see figure 5). After this moment the function $u(x,t)$ reaches its maximum

$$u_{max}(t) = \frac{1}{k}\left(\frac{R}{x_{max}}\right)^{n-1}\frac{\mathrm{d}s}{\mathrm{d}t} \tag{53}$$

at the point
$$x_{max}(t) = \left[\frac{nh}{Rk(n-1)}\right]^{1/n}(1-\sigma)^{1/n}R \tag{54}$$

which is inside the contact region, $0 < x_{max}(t) < c(t)$, and tends to the centre point $x_{max} \to 0$, as $t \to T$. We will have $u_{max} = c_0$, where c_0 is the sound velocity in the liquid at rest, when

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\tau} = \frac{k}{M}\left[\frac{nh}{Rk(n-1)}\right]^{(n-1)/n}(1-\sigma)^{(n-1)/n}, \tag{55}$$

where $M = V_0/c_0$ is the Mach number. This condition is not necessarily satisfied for arbitrary values of the parameters and must be verified in the numerical calculations. The present scheme is valid only when $u_{max}/c_0 \ll 1$.

The pressure distribution over the wetted part of the entering contour is given by

$$p(x,t) = \tfrac{1}{2}\rho\frac{c^2}{h^2}s'^2 P\left(\frac{x}{c},\sigma\right), \tag{56}$$

$$P(z,\sigma) = 2\int_z^1 \frac{\xi\,\mathrm{d}\xi}{(\xi^n(U+\sigma)+1-\sigma)^2} + 2\frac{s''h}{s'^2}\int_z^1 \frac{\xi\,\mathrm{d}\xi}{\xi^n(U+\sigma)+1-\sigma}$$
$$-\frac{z^2}{(z^n(U+\sigma)+1-\sigma)^2} + \frac{1}{(U+1)[1-(U+1)^{-1/2}]}, \tag{57}$$

where $0 \leqslant z \leqslant 1$, $0 \leqslant \sigma \leqslant 1$. For a parabolic shape entering with a constant velocity, $n = 2$, $s(t) = V_0 t$, we obtain

$$p(x,t) = \rho V_0^2 \frac{R}{h} G\left(\frac{x}{c}, \sigma\right), \quad c = (2hR)^{1/2}(U+\sigma)^{1/2}, \tag{58}$$

$$G(z,\sigma) = (U+\sigma)\left(\frac{1+(U+1)^{-1/2}}{U(U+1)} - \frac{1}{z^2(U+\sigma)+1-\sigma} - \frac{z^2}{(z^2(U+\sigma)+1-\sigma)^2}\right). \tag{59}$$

It is interesting to note that

$$c(t) \approx (6RV_0 t)^{1/2}, \tag{60}$$

$$p(x,t) \approx 3\rho V^2 R/h \tag{61}$$

as $t \to 0$. It can be shown that $p(x,t)$ reaches its maximum value at the centre of the contact region and $p = O[(1-\sigma)^{-1}]$ as $\sigma \to 1$. This means that at the final stage of the impact the liquid compressibility is of importance and the flow pattern will be more complicated than that presented here.

7. Conclusions

At the initial stage of the shallow-water impact the Wagner theory can be used. Then the presence of the bottom becomes important and at this stage the depth of the body penetration can be approximately disregarded in comparison with both the contact region dimension and the liquid layer depth. At the next stage, which is the subject of the present paper, the variation of the liquid domain with time cannot be neglected and, moreover, it is the main factor of the process. At the final stage, when the entering body is close to the bottom, the liquid compressibility and the possibility of soliton formation must be taken into account. But the duration of this stage is small and can be neglected to a first approximation.

The problem considered is quite suitable for experimental analysis, because the duration of the process is relatively large and one does not need modern equipment to follow the liquid flow. Preliminary experimental results by V. I. Bukreev (1994, personal communication) on a blunt-wedge impact on shallow water show that the main effect is the spray jet formation. The spray jet is strong: its thickness is comparable with the depth of the liquid layer. A soliton is not formed: most of the liquid displaced by the entering body leaves the liquid layer as spray jets.

The present analysis can be extended to the axisymmetrical problem. In the three-dimensional case as well as for the oblique-entry problem, some difficulties may be present in obtaining the solution in region I only.

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